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INTRINSIC METRIC AND STANDARDNESS FOR THE INFINITE GRADED GRAPHS

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5.Uniform compactness and standardness of graph and projective limit.

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8.Problems.

Graded graphs

Let $\Gamma = \bigsqcup_{n \in \mathbb{N}} \Gamma_n$ is a $\mathbb{N}$-graded graph with finite levels Γ_n, and edges between the vertices of the adjacent levels. See examples on the next slides. $\Gamma_0 = \{1\}$

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EXAMPLES:

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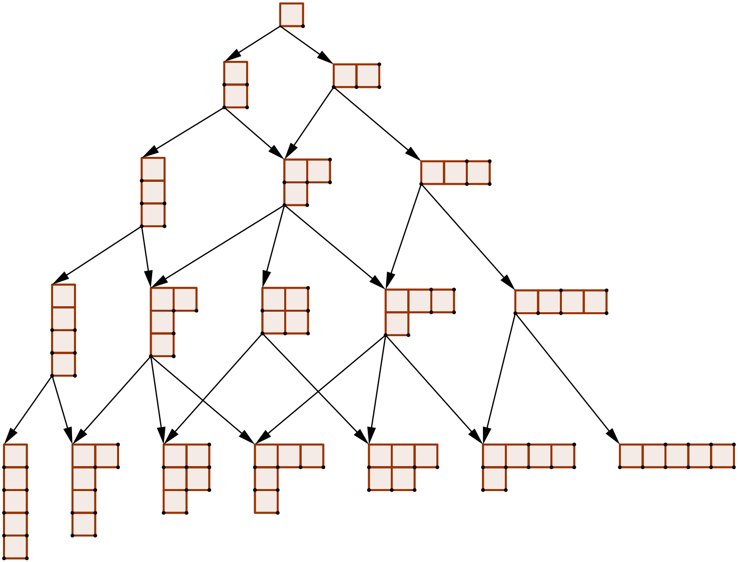

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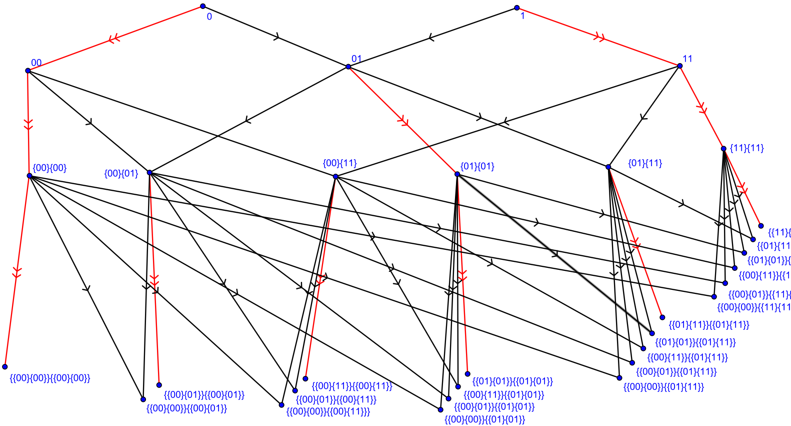
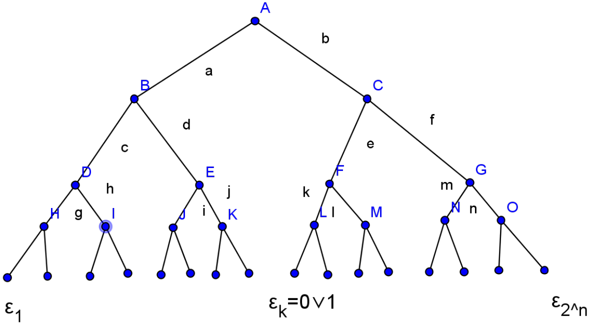


Orbits



Qube of dimension 2^n (vectors with 2^{2^n} coordinats)

Graph NUP is the graph of the orbits of the group of automorphisr
of the tree which acts on the Qube

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Definition

A probability measure μ on the space of paths $T(\Gamma)$ called Λ-invariant if the conditional probability of the measure on the subspace of $T(\Gamma)$ of pathes which contain edge (u, v), is equal to λ_u^v.

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There is a canonical bijection between central measures and set of traces on the algebra $C(\Gamma)$ corresponded to the graph Γ. Algebra $C(\Gamma)$ is locally semi-simple algebra for which graph Γ is graph of simple modules of it. If $C(\Gamma)$ is group algebra of locally finite group, then central measures are characters of the group. Ergodic central measures correspond to indecomposable traces or characters.

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Problem: To describe the projective limit of the simplecies.
The same question can be put for general set of the cotransition probabilities Λ.
Our main result is to divide the set of graded graphs on to two classes depending on the type of answer on the problem: -standard when the problem has visible answer and nonstandard.

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Let Σ is a simplex of the dimension n. For $x \in \Sigma$ denote $\mu_x^1, \mu_x^2 \dots \mu_x^{n+1}$ its the barycenter coordinates of the point x and μ_x — the corresponding measure on $ex(\Sigma)$ -set of vertices of Σ, so the barycenter of measure μ is x : $\mathrm{bar}(\mu_x) = x$.

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We will use the bijection $x \dashrightarrow \mu_x$ between points of Σ and probability measures on $ex(\Sigma)$.

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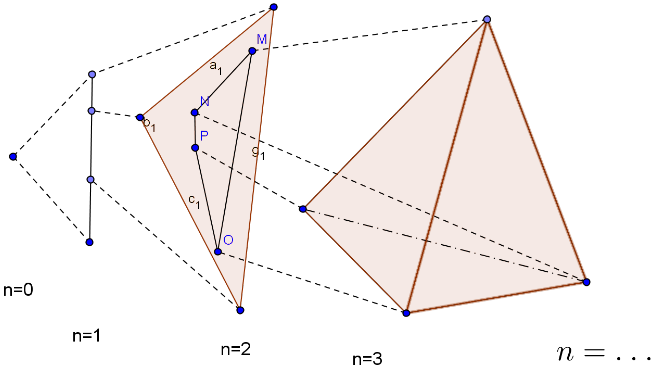

$d \geq n$

$n = \infty - ???$

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Theorem

Then point $\{x_n\}$ is extremal point of Σ_∞ iff

$$\forall m \quad w - \lim_n p^*_{n,m}(\mu_{x_n}) = \delta_{x_m}.$$

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How to extend it to the whole simplex?
Let $X, Y \in \Sigma$, $ex(\Sigma) = \{e_1, \dots e_k\}$ and r is metric on $ex(\Sigma)$; then

$$\rho(X, Y) \doteq k_r(\mu_X, \mu_Y),$$

where $k_r(.,.)$ is Kantorovich (transport) metric on the measures on $(ex(\Sigma), r)$.

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Formula:

$$\rho(X, Y) \equiv k_r(\nu_X, \nu_Y) = \min_{\psi} \sum_{i,j} \Psi_{i,j} r(e_i, e_j) :$$

$$\Psi = \{\psi_{i,j} \geq 0\}; \quad \sum_j \Psi_{i,j} = \mu_X(i), \sum_i \Psi_{i,j} = \mu_Y(j).$$

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(lemma P-V)

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(lemma P-V)
Now we want to transfer metric from the level Γ_n to the level Γ_{n+1} and from the simplex to the next one. Because of λ structure each vertex of on Γ_{n+1} is the measure on Γ_n, consequently, point of Σ_n.

Transfer of metric in bi-partive graphs

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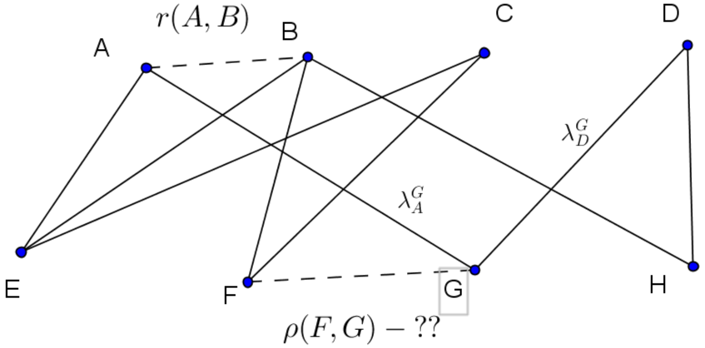
metric.png



The metric r on the set A, B, C, D done

The probabilities λ are known

To define a metric on the set E, F, G, H

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and initial metric ρ_1 on the Σ_1,
then we can sequentially define a sequence of the metrics ρ_n on the simplices $\Sigma_n, n > 1$: we consider metric ρ_n on $p_n(ex\Sigma_{n+1}) \subset \Sigma_n$, thus we have a metric on the vertices of Σ_{n+1} and then ρ_{n+1} is extension of it on whole simplex Σ_{n+1}.

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Important Remark. There is no serious dependence of the properties of intrinsic metric on initial metric ρ_1.

Standardness

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Graded graph $\Gamma = \bigcup \Gamma_n$ (and the corresponded projective limit of simplecies $\lim_n \Sigma(\Gamma_n) \equiv \Sigma_\infty(\Gamma)$) called standard if the set of all simplecies $\Sigma_n, n = 1 \dots \infty$ are uniformly compact,

Standardness

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Graded graph $\Gamma = \bigcup \Gamma_n$ (and the corresponded projective limit of simplecies $\lim_n \Sigma(\Gamma_n) \equiv \Sigma_\infty(\Gamma)$) called standard if the set of all simplecies $\Sigma_n, n = 1 \dots \infty$ are uniformly compact, this means:

$$\forall \epsilon > 0, \sup_n \{N_\epsilon(\Sigma_n, \rho_n)\} < \infty,$$

here $N_\epsilon(K, r)$ is number of points in the ϵ-net of compact K with respect to the metric r.

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Theorem

If the projective limit is standard then the intrinsic metrics ρ_n define intrinsic metric on the limit simplex Σ_{infty}, and the last is compact so weak topology agrees with extended intrinsic metric.

Corollaries

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Conclusion. The usual difficulties in the problem of these type is the following: the list of the invariant (central) measures is "known" and the goal is to prove that indeed this is the list. Usually this is difficult question because procedure of the finding of the measures like "ergodic method" is too complicate. Our method looks like "apriority estimation": if we prove the standardness then the visual list is complete.

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4.Limit shape theorems.

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Ergodic measures (extremal points of the projective limit).
The Orbits of the group of automorphism of the dyadic tree.
2.Young graph and other: how to prove the standardness.